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# Wilson Loop Invariants from $W_N$ Conformal Blocks

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**ABSTRACT:** Knot and link polynomials are topological invariants calculated from the expectation value of loop operators in topological field theories. In 3D Chern-Simons theory, these invariants can be found from crossing and braiding matrices of four-point conformal blocks of the boundary 2D CFT. We calculate crossing and braiding matrices for  $W_N$  conformal blocks with one component in the fundamental representation and another component in a rectangular representation of  $SU(N)$ , which can be used to obtain HOMFLY knot and link invariants for these cases. We also discuss how our approach can be generalized to invariants in higher-representations of  $W_N$  algebra.

**KEYWORDS:** Conformal Field Theory,  $W_N$  algebra, Knot Theory, Topological Quantum Field Theory

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## 1 Introduction

In the late 80's, the important connection between quantum field theory (QFT) and Jones Polynomials [1] was uncovered by Edward Witten [2] showing that knot theory is deeply connected to topological QFTs. The particular example studied by Witten was  $SU(N)$  Chern-Simons theory (CS) in a 3D compact manifold  $M$ . Non-trivial states in CS are topological as there are no dynamical degrees of freedom in this theory. Therefore, Wilson loops are natural elementary states of CS. To construct a non-trivial state in this sense, we must create knotted loops or link several loops together. The expectation value of these composite objects gives us certain topological invariants called *knot (or link) polynomials*. For the  $SU(2)$  case, Witten has shown that one obtains Jones polynomials from the Wilson loop expectation values. This relation between Chern-Simons and knot theory is an important example of integrability in quantum field theories, which serves as a tool to organize and construct physical theories.

The explicit construction of these invariants starts with a partition of  $M$  into two manifolds with a boundary. To each boundary is associated a WZNW theory whose Hilbert space is the space of conformal blocks with Wilson lines as punctures on the boundary. Braiding matrices of the boundary CFT can then be used to construct the original knot or link in  $M$  [2–5]. Here we call this the *crossing-matrix method*. This approach can be related to quantum groups, as the crossing transformations of conformal blocks are directly related to  $SU(N)_q$  quantum Racah matrices [6–9]. For the most recent and broad discussion of this method, see [10]. When combined with the evolution method [11] and cabling procedure [12], the crossing matrices can be extrapolated to give explicit formulas for many families of knots and links [13]. This combination of techniques gives not only Jones polynomials  $J_R^C(q)$  [14] but also can be uplifted to calculate HOMFLY polynomials  $H_R^C(q, A)$  [15, 16] and *superpolynomials*

$P_R^C(q, A, t)$ . Superpolynomials appeared in physics in the connection between topological string theories, M-theory and Chern-Simons [17–19]. Understanding better how HOMFLY polynomials come about in  $W_N$  models might also shed some light on the nature of superpolynomials. The crossing-matrix method has also been used to construct loop operators of  $\mathcal{N} = 2$  gauge theories via its AGT relation with Liouville [20, 21] and Toda field theory [22–24], so the study of  $W_N$  conformal blocks is also interesting for this AGT approach.

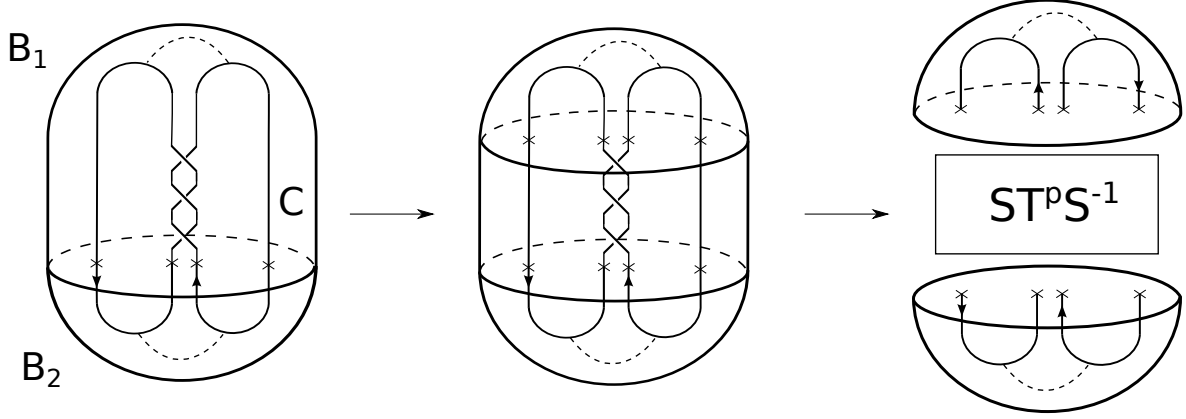
Going back to the relation between knots and conformal blocks, the crossing-matrix method is well-understood in the Virasoro case ( $W_2$  algebra), but has not been explicitly developed before for  $W_N$  algebras to the authors knowledge. Here we develop the crossing-matrix approach for  $W_N$  minimal models directly from the CFT point of view in detail. Evidence has been put forward in [25] that knot and link invariants in  $W_N$  models should factorize in terms of  $SU(N)_q$  invariants, but the crossing matrices have not been calculated. Much more is known about  $SU(N)_q$  Racah matrices and topological invariants constructed with it [26, 27], including representations with non-trivial multiplicities [28]. Therefore, as proposed in [25], we expect that  $W_N$  invariants should reduce to  $SU(N)_q$  invariants and indeed that is what we find in the cases studied below. However, we still have limited information about higher-representations and it is not clear if the crossing-matrices will factorize in general for  $W_N$  correlators.

Four-point Virasoro conformal blocks need only one completely degenerate field to obey a hypergeometric differential equation, the BPZ equation [29]. However, for higher  $W_N$  algebras ( $N > 2$ ), we need one more constraint to find a differential equation and to obtain explicit crossing  $S$  and braiding  $T$  matrices [30, 31]. If we also set some other field to be semi-degenerate, the conformal blocks obey a generalized hypergeometric equation. In this note, we construct  $S$  and  $T$  matrices with two fields in the fundamental and anti-fundamental representations of  $SU(N)$  and the other two in a rectangular representation and its conjugate. These cases are somewhat degenerate with respect to higher-representations of  $W_N$  primary fields, as the dimension of the space of conformal blocks is two-dimensional, but are the first step to obtain more general S-matrices for higher-representations [14, 32]. In our particular case, the generalized hypergeometric equation reduces to a Gauss hypergeometric equation, for which the connection formulas are explicitly known and, thus, the crossing matrices.

In section 2, we revise the relation between Wilson loop operators in 3D Chern-Simons, knot invariants and conformal blocks. In section 3, we set up our notation by reviewing how to obtain knot invariants in Virasoro models. In section 4, we discuss how to calculate knot and link invariants from  $W_N$  conformal blocks. Finally, in section 5 we present our conclusions and discuss further developments.

## 2 Knot and Link Invariants from Conformal Blocks

Following Witten’s construction [2], we are interested in calculating the expectation value of non-trivial Wilson loops forming a knot or link  $C$  embedded in a closed three-dimensional manifold  $M$  in Chern-Simons theory. For simplicity, we take  $M$  isomorphic to a 3-sphere. We



**Figure 1.** Splitting a two-bridge knot  $C$  into an inner product of conformal blocks. The braiding and fusion matrices allow for a reconstruction of  $C$  from basic conformal blocks with zero weight intermediate states.

can then cut  $C$  into two bounded parts,  $B_1$  and  $B_2$ , by slicing  $M$  with a 2-dimensional surface (see fig. 1). To each  $B_k$  we relate a state  $\psi_k$  of the WZNW CFT defined on its boundary  $\partial B_k$ . In this interpretation, a Wilson loop invariant is given by the inner product between these two states

$$Z_R(C) = \left\langle \text{Tr}_R \mathcal{P} \exp \left( \oint_C A \right) \right\rangle_{CS} = \langle \psi_1 | \psi_2 \rangle. \quad (2.1)$$

The Hilbert space  $\mathcal{H}_k$  of each  $B_k$  is isomorphic to the space of conformal blocks of the boundary CFT. These blocks have extra proportionality parameters coming from the braiding and crossing operations to build up  $C$ , as explained below. Here and in the rest of the paper we restrict our attention to invariants build up from four-point conformal blocks, also called *two-bridge* states.

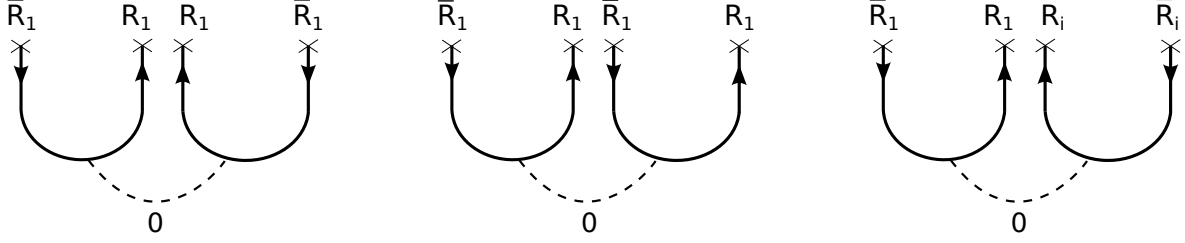
The two-bridge knot invariants can be constructed via braiding and closure of a  $j$ -channel conformal block  $\mathcal{F}_0^j$  with zero weight<sup>1</sup> intermediate state, where the index  $j$  represents either the s, t or u-channel. Each puncture represented in fig. 1 corresponds to a field insertion of the conformal block in some representation  $R$  of  $SU(N)$ , as shown in fig. 2. We fuse the relevant fields via a crossing matrix  $S$  and then braid several times with a diagonal half-monodromy matrix<sup>2</sup>  $T$ . General  $S$  and  $T$  matrices depend on the field representations

$$S_{i_1 i_2} \begin{bmatrix} R_2 & R_3 \\ R_1 & R_4 \end{bmatrix}, \quad T_{j_1 j_2} [R_1 \ R_2], \quad (2.2)$$

with the internal indices being labeled by the result of the fusion of appropriate representations, i.e.,  $i_1, i_2 \in (R_1 \otimes R_2) \cap (R_3 \otimes R_4)$  and  $j_1, j_2 \in [R_k]$ , where  $[R_k]$  represents the

<sup>1</sup>As the total charge of a closed manifold must be zero.

<sup>2</sup>These matrices are also called fusion matrix  $F$  and braiding matrix  $B$  in the literature.  $S$  and  $T$  usually refers to modular transformations on the torus.



**Figure 2.** Different possibilities of basic conformal blocks. From left to right: parallel, antiparallel and mixed case.

intermediate states in the appropriate channel. Non-trivial multiplicities might also appear in the fusion rules of certain fields but we do not consider those here. In the following, we omit the matrix dependence on representations.

Back to figure 2, lines going up in representation  $R_j$  must close with lines going down in the conjugate representation  $\bar{R}_j$  after the braiding evolution in the last step of fig. 1. The first two cases in fig. 2 have parallel and anti-parallel fusing strands, respectively. The third case has one of the bridges in a different representation. Only in the first two cases we can close the strands to form a knot<sup>3</sup> or a link and in the third case we have only links. In this paper, we are going to consider the first two cases with  $R_1$  in the fundamental representation of  $SU(N)$  and the third case with  $R_i$  in a rectangular representation of  $SU(N)$ .

For different sequences made up of  $S$  and  $T$  matrices, we can construct several types of knots [10, 14]. The simplest examples of two-bridge invariants are described by the following formula

$$Z_R^{j,p}(C) \equiv \langle \mathcal{F}_0^j | ST^p S^{-1} | \mathcal{F}_0^j \rangle = (ST^p S^{-1})_{00}, \quad p \in \mathbb{Z}^+, \quad (2.3)$$

where the last equality represents the singlet diagonal component of the matrix. When  $p$  is odd, we have a knot, and when it is even, we have a link. In the  $SU(N)_q$  case, these invariants are proportional to HOMFLY polynomials depending on the variables  $q = e^{\pi i/(k+N)}$  and  $A = q^N$  [26–28]. The proportionality factor depends on the choice of framing for the Wilson loops, but are canonically chosen to not depend explicitly on  $N$ , except through  $A$  [28]. When  $N = 2$ , we get Jones polynomials as a special case of the HOMFLY ones.

In this paper, we look for the appropriate  $S$  and  $T$  matrices for fields labeled by representations of  $W_N$  algebra. Explicit calculation shows that these matrices are not properly normalized to give the usual quantum Racah matrices [28, 33, 34]. In order to have an explicit representation of the braid group, we have to find a conformal block normalization such that  $S$  is a unitary hermitian matrix, that is,

$$SS^\dagger = \mathbb{1}, \quad S = S^\dagger, \quad \Rightarrow \quad S^2 = \mathbb{1}. \quad (2.4)$$

This property will allow us to fix the normalization. The  $S$  matrix is not hermitian in general, but it will be valid in our particular two-dimensional case. For knots, we have two

<sup>3</sup>In the second case, we can have twist knots [14].

types of crossing and braiding matrices corresponding to the parallel case  $(S, T)$  and to the antiparallel case  $(\bar{S}, \bar{T})$ . We also need that the Yang-Baxter equation be satisfied for certain 3-strand moves. These matrices must obey the unknot constraint [10]

$$S\bar{T}\bar{S} = T^{-1}S\bar{T}^{-1}, \quad (2.5)$$

and the Yang-Baxter equation

$$S\bar{T}\bar{S}\bar{T}STS = T\bar{S}\bar{T}\bar{S}\bar{T}. \quad (2.6)$$

These two equations will then allow us to choose a correct framing for the  $T$  matrices below.

## 2.1 Normalization of Conformal Blocks

In general, a four-point correlation function of primary fields in a CFT with symmetry algebra  $g$  can be written as

$$\langle V_{\alpha_1}(z)V_{\alpha_2}(1)V_{\alpha_3}(0)V_{\alpha_4}(\infty) \rangle = (\mathcal{G}^s)^\dagger \mathcal{M}^s \mathcal{G}^s = (\mathcal{G}^t)^\dagger \mathcal{M}^t \mathcal{G}^t, \quad (2.7)$$

with  $\alpha_i$  being  $g$ -valued vectors labelling the primaries,  $\mathcal{G}^k = \mathcal{G}^k(z)$  are the conformal blocks in the  $k = s, t$ -channels and  $\mathcal{M}^k$  are constant matrices (more generally, bilinear transformations) formed by the product of structure constants of each channel. The space of conformal blocks  $\mathcal{H}^k$  is finite dimensional depending if one or more fields in (2.7) are degenerate. This is always the case for rational conformal field theories. In the following, we suppose that the matrices  $\mathcal{M}^k$  are diagonal, which is not necessarily true in general because of non-trivial monodromy properties of higher-spin conformal blocks [35]. We shall review the calculation of  $W_N$  correlators in the next section.

We want to change the normalization of the conformal blocks in such a way that the  $S$  matrix is an unitary hermitean matrix. First, let us define the new blocks as  $\mathcal{F}^k = N_k \mathcal{G}^k$  (with no index summation), where  $k = s, t$  denotes the respective channels and  $N_k$  are diagonal normalization matrices. Also, we have that  $\mathcal{G}^s = S \mathcal{G}^t$  and, using this in (2.7), we get

$$S^\dagger \mathcal{M}^s S = \mathcal{M}^t, \quad (2.8)$$

Changing the  $S$  matrix to  $\tilde{S}$  in the new normalization, we write  $S = N_s^{-1} \tilde{S} N_t$ . If we plug this into (2.8) we obtain

$$\mathcal{M}^k = \alpha N_k^\dagger N_k, \quad \alpha \in \mathbb{R}. \quad (2.9)$$

In the following, we set  $\alpha = 1$  as it depends only on the overall normalization of the correlation function.

Now, as will be clear below, we suppose that  $\mathcal{H}^k$  is two-dimensional. Then we can

parametrize the normalization matrices as

$$N_k = \delta_k \begin{pmatrix} \zeta_k^{-1} & 0 \\ 0 & \zeta_k \end{pmatrix}. \quad (2.10)$$

Setting  $\mathcal{M}^k = \text{diag}(C_1^k, C_2^k)$ , we get

$$\frac{C_1^k}{C_2^k} = \frac{1}{|\zeta_k|^4}, \quad (2.11)$$

where  $C_j^k$  corresponds to the products of structure constants in the  $k$ -channel appearing in (2.7). We determine  $\delta_k$  up to a phase by

$$\text{Tr } \mathcal{M}_k = C_1^k + C_2^k = |\delta_k|^2 (|\zeta_k|^{-2} + |\zeta_k|^2). \quad (2.12)$$

From equations (2.11) and (2.12) we thus get

$$|\zeta_k| = \left( \frac{C_2^k}{C_1^k} \right)^{1/4}, \quad |\delta_k| = \left( C_1^k C_2^k \right)^{1/4}. \quad (2.13)$$

Therefore, if we can find the products of structure constants  $C_j^k$ , we can fix the normalization. When we know the  $S$  matrix, we can go the other way around and use it to find the structure constants, which is at the core of the bootstrap approach.

The discussion presented here should be compared with [34], where its authors define the proper normalization of conformal blocks to obtain the Racah-Wigner  $b$ -6j symbols associated to the modular double of  $\mathcal{U}_q(\text{sl}(2, \mathbb{R}))$  with  $q = e^{i\pi b^2}$ . The representations of this quantum group will explicitly appear below in our approach of loop invariants in the Virasoro case.

## 2.2 Knot and Link Invariants from Virasoro Representations

Here we briefly review how to calculate Jones polynomials from  $S$  and  $T$  matrices related to fields in Virasoro representations. We start with a few definitions following [36], for example. Chiral vertex operators  $V_\alpha$  are labeled by charge vectors  $\alpha \equiv \alpha_{r,s}$ , which also label the conformal dimension of  $V_\alpha$

$$\Delta_{(r,s)} = \alpha_{r,s}(Q - \alpha_{r,s}), \quad \alpha_{r,s} = \frac{1}{2} \{ (1-r)b + (1-s)b^{-1} \}, \quad (2.14)$$

where  $Q = b + 1/b$ . The integers  $r, s$  label unitary irreducible Verma modules with central charge  $c$  less than one, appearing in the minimal models. Let us calculate the conformal blocks of the 4-point correlation function holomorphic part

$$F_{\hat{\alpha}}(z, z_1, z_2, z_3) \equiv \langle V_\alpha(z) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle, \quad (2.15)$$

where  $\hat{\alpha} = (\alpha, \alpha_1, \alpha_2, \alpha_3)$  and the field  $V_\alpha$  is degenerate at level 2, i.e.,  $(r, s) = (1, 2)$  or  $(2, 1)$ . The charge vectors must obey the neutrality condition

$$\alpha + \sum_{i=1}^3 \alpha_i + mb + nb^{-1} = Q, \quad m, n \in \mathbb{Z}_+. \quad (2.16)$$

One of the fields being degenerate at level 2 implies either one of the null vector conditions below

$$(L_{-2} + b^2 L_{-1}^2) V_\alpha = 0 \quad \text{for} \quad (r, s) = (1, 2), \quad (2.17)$$

$$(L_{-2} + \frac{1}{b^2} L_{-1}^2) V_\alpha = 0 \quad \text{for} \quad (r, s) = (2, 1). \quad (2.18)$$

Let us focus on the choice  $(r, s) = (2, 1)$ , such that  $\alpha = -\frac{b}{2}$ . Using the conformal Ward identity, we find the action of the Virasoro operators on the correlators, which then implies in the BPZ equation

$$\left( \frac{1}{b^2} \frac{\partial^2}{\partial z^2} + \sum_{i=1}^3 \left[ \frac{1}{z - z_i} \frac{\partial}{\partial z_i} + \frac{\Delta_i}{(z - z_i)^2} \right] \right) F_{\hat{\alpha}}(z, z_1, z_2, z_3) = 0. \quad (2.19)$$

Using  $SL(2, \mathbb{C})$  invariance, we set  $(z_1, z_2, z_3) \rightarrow (0, \infty, 1)$  and then get

$$\frac{1}{b^2} F_{\hat{\alpha}}''(z) + \frac{2z - 1}{z(1 - z)} F_{\hat{\alpha}}'(z) + \left( \frac{\Delta_1}{z^2} + \frac{\Delta_3}{(1 - z)^2} + \frac{\Delta_{(2,1)} + \Delta_1 - \Delta_2 + \Delta_3}{z(1 - z)} \right) F_{\hat{\alpha}}(z) = 0. \quad (2.20)$$

Finally, we put this equation into hypergeometric form by making  $F(z) = z^{b\alpha_1} (1 - z)^{b\alpha_3} G(z)$

$$\{z(1 - z) \frac{\partial^2}{\partial z^2} + [C - (A + B + 1)z] \frac{\partial}{\partial z} - AB\} G(z) = 0, \quad (2.21)$$

where

$$A = \frac{1}{2} + b(\alpha_1 + \alpha_3 - Q) + b(\alpha_2 - \frac{Q}{2}), \quad (2.22)$$

$$B = \frac{1}{2} + b(\alpha_1 + \alpha_3 - Q) - b(\alpha_2 - \frac{Q}{2}), \quad (2.23)$$

$$C = 1 + b(2\alpha_1 - Q). \quad (2.24)$$

The solutions of (2.21) are given by hypergeometric functions  ${}_2F_1(A, B; C|z)$ . If we label the conformal blocks as  $\mathcal{F}^k = (F_1^k \ F_2^k)$ , where  $k = s, t$  denotes the channels, we have the s-channel conformal blocks

$$F_1^s(z) = z^{b\alpha_1} (1 - z)^{b\alpha_3} {}_2F_1(A, B; C|z), \quad (2.25a)$$

$$F_2^s(z) = z^{b(Q - \alpha_1)} (1 - z)^{b\alpha_3} {}_2F_1(A - C + 1, B - C + 1; 2 - C|z), \quad (2.25b)$$



and the  $t$ -channel conformal blocks

$$F_1^t(z) = z^{b\alpha_1} (1-z)^{b\alpha_3} {}_2F_1(A, B; A+B-C+1 | 1-z), \quad (2.26a)$$

$$F_2^t(z) = z^{b\alpha_1} (1-z)^{b(Q-\alpha_3)} {}_2F_1(C-A, C-B; C-A-B+1 | 1-z). \quad (2.26b)$$

The crossing matrix  $S$  is given by  $\mathcal{F}^s = S\mathcal{F}^t$  where

$$S = \begin{pmatrix} \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} & \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} \\ \frac{\Gamma(2-C)\Gamma(C-A-B)}{\Gamma(1-A)\Gamma(1-B)} & \frac{\Gamma(2-C)\Gamma(A+B-C)}{\Gamma(A-C+1)\Gamma(B-C+1)} \end{pmatrix}. \quad (2.27)$$

In the particular case of the other fields also in the fundamental representation,  $\alpha_1 = \alpha_3 = -b/2$  and  $\alpha_2 = Q - \alpha_1$ . Thus, we get

$$S = \begin{pmatrix} \frac{1}{[2]} & \frac{\Gamma(-2b^2)\Gamma(-2b^2-1)}{\Gamma(-3b^2-1)\Gamma(-b^2)} \\ \frac{\Gamma(2b^2+1)\Gamma(2b^2+2)}{\Gamma(b^2+1)\Gamma(3b^2+2)} & -\frac{1}{[2]} \end{pmatrix} = \begin{pmatrix} \frac{1}{[2]} & \frac{\sqrt{[3]}}{[2]} \zeta^2 \\ \frac{\sqrt{[3]}}{[2]} \zeta^{-2} & -\frac{1}{[2]} \end{pmatrix} = U \begin{pmatrix} \frac{1}{[2]} & \frac{\sqrt{[3]}}{[2]} \\ \frac{\sqrt{[3]}}{[2]} & -\frac{1}{[2]} \end{pmatrix} U^{-1} \quad (2.28)$$

where  $q = e^{i\pi b^2}$ ,  $[N] = \frac{q^N - q^{-N}}{q - q^{-1}}$  and

$$U = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \zeta^2 = \frac{\sqrt{[3]}}{[2]} \frac{\Gamma(b^2+1)\Gamma(3b^2+2)}{\Gamma(2b^2+1)\Gamma(2b^2+2)}. \quad (2.29)$$

It is now easy to check that  $S^2 = \mathbb{1}$ , as required by (2.4). The  $s$ -channel braiding matrix  $T$  is obtained by making a half turn around zero in (2.25)

$$T = N(q) \begin{pmatrix} q^{-1/2} & 0 \\ 0 & -q^{3/2} \end{pmatrix}, \quad (2.30)$$

where  $N(q)$  is an overall normalization. The Yang-Baxter equation in this case is  $(ST)^3 = \mathbb{1}$  and this let us choose  $N(q) = -q^{-1/2}$ . Summing up, the matrices representing the braid group  $B_3$  colored with  $SU(2)$  representations are

$$S = \begin{pmatrix} \frac{1}{[2]} & \frac{\sqrt{[3]}}{[2]} \\ \frac{\sqrt{[3]}}{[2]} & -\frac{1}{[2]} \end{pmatrix}, \quad T = \begin{pmatrix} -\frac{1}{q} & 0 \\ 0 & q \end{pmatrix}. \quad (2.31)$$

We can construct knot and link polynomials by starting with the  $t$ -channel conformal blocks, braiding representations in the  $s$ -channel  $k$  number of times and then going back to the

$t$ -channel, like in fig. 1. In this way, we get

$$ST^p S = \frac{1}{[2]^2} \begin{pmatrix} q^p[3] + (-1)^p q^{-p} & - (q^p + (-1)^{p+1} q^{-p}) \sqrt{[3]} \\ - (q^p + (-1)^{p+1} q^{-p}) \sqrt{[3]} & q^p + (-1)^p q^{-p} [3] \end{pmatrix}. \quad (2.32)$$

According to the fusion rules, the conformal block with zero intermediate weight is (2.25b) and, thus, the knot/link invariant of interest is the second diagonal component of (2.32), which is an *unreduced* Jones invariant. The reduced Jones knot polynomial is defined as the ratio of the unreduced polynomial and the *unknot* in the same representation

$$J_R^{C,2k+1}(q) = \frac{Z_R^t(2k+1|C)}{Z_R^t(1|C)} = \frac{(ST^{2k+1}S)_{22}}{(STS)_{22}} = -q^4 + q^2 + q^{-2}. \quad (2.33)$$

Notice that the second diagonal component can be recovered from the first by making  $q \rightarrow -1/q$ . A more extensive discussion of the types of knots and link invariants calculated in this way is given in [10, 14].

### 3 Toda Field Theory and $W_N$ Conformal Blocks

In this section, we review the machinery of Toda field theory, discussed in [37], in order to generalize the construction above for  $S$  and  $T$  matrices in  $W_N$  models. Specifically, we fix two fields in the four-point function to be in the fundamental representation of  $SU(N)$  paired with its conjugate representation and deduce some results for the other fields in an more general representation. This section follows mostly the definitions and conventions of [37]. Other relevant references about correlators in  $W_N$  models are [35, 38].

The generalization of Liouville theory extending Virasoro to  $W_N$  algebra is called Toda field theory (TFT). The basic field is a scalar field  $\varphi = \sum_{i=1}^{N-1} \varphi_i e_i$ , where  $e_i, i = 1, \dots, N-1$ , are the simple roots of  $su(N)$  algebra. The most important information about the algebra is contained in the *Cartan matrix*,  $K_{ij}$ , defined by the inner product of the simple roots,  $K_{ij} = (e_i, e_j)$ . From the inner product, one can define the dual weight space in terms the *fundamental weights*  $\omega_k$  by  $(\omega_k, e_j) = \delta_{kj}$  and the *quadratic form* of the algebra by  $(\omega_i, \omega_j) = K_{ij}^{-1}$ . A highest weight  $\lambda$  takes the form

$$\lambda = \sum_{i=1}^{N-1} \lambda_i \omega_i, \quad (3.1)$$

where  $(\lambda_1, \dots, \lambda_{N-1}) \in \mathbb{Z}_{\geq 0}^{N-1}$  are called Dynkin labels. In particular, the conjugate representation of  $\lambda$  is represented by  $\bar{\lambda} = \sum_{i=1}^{N-1} \lambda_{N-i} \omega_i$ .

To each highest weight we can associate a *partition*  $\lambda = \{\ell_1; \ell_2; \dots; \ell_{N-1}\}$  where  $\ell_i = \lambda_i + \lambda_{i+1} + \dots + \lambda_{N-1}$ . We then associate a *Young tableau* to the partition by assigning  $\ell_i$

boxes to the  $i$ -th row of the tableau. Some simple examples are

$$\begin{aligned} F_1 &= (1, 0, 0, \dots, 0) \sim \square \\ F_2 &= (2, 0, 0, \dots, 0) \sim \square \square \\ A_2 &= (0, 1, 0, \dots, 0) \sim \begin{array}{c} \square \\ \square \end{array} \end{aligned}$$

where the first example is the fundamental representation, the second a symmetric representation and the third an antisymmetric representation. Young diagrams are useful to build up tensor product representations and thus analyze possible states of fusion rules. For more details, see [39], for example. To find all the states in an irreducible module with highest weight  $\lambda$ , we subtract all possible combinations of simple roots  $e_i$  up to  $\lambda_i e_i$  for each positive  $\lambda_i$ . Then we repeat the process with the new weights until there is no way to produce a new weight with positive Dynkin label. In the case of the fundamental representation, the weights are expressed as

$$h_k = \omega_1 - \sum_{i=1}^{k-1} e_i, \quad k = 1, \dots, N. \quad (3.2)$$

The TFT action on a Riemann surface with reference metric  $\hat{g}_{ab}$  and scalar curvature  $\hat{R}$  is given by

$$S_{TFT} = \int \left( \frac{1}{8\pi} \hat{g}^{ab} (\partial_a \varphi, \partial_b \varphi) + \frac{(Q, \varphi)}{4\pi} \hat{R} + \mu \sum_{k=1}^{N-1} e^{b(e_k, \varphi)} \right) \sqrt{\hat{g}} d^2 x, \quad (3.3)$$

where  $\mu$  is the cosmological constant and  $Q$  is the background charge. To ensure conformal invariance, we must set the charge to be

$$Q = (b + 1/b)\rho, \quad \rho = \sum_{k=1}^{N-1} \omega_k, \quad (3.4)$$

where  $\rho$  is the *Weyl vector* of the algebra. The theory is invariant under symmetries generated by the currents  $W^k(z)$  with spins  $k = 2, 3, 4, \dots, N$  and its antiholomorphic counterparts. The current  $W^2(z) \equiv T(z)$  is equal to the energy-momentum tensor and the other are higher-spin currents. Together, these currents generate the  $W_N$  algebra containing the Virasoro algebra with central charge  $c = N - 1 + 12Q^2$ .

The Toda correlators can be calculated in the Coulomb gas formalism by introducing the chiral vertex operators  $V_\alpha = e^{(\alpha, \varphi)}$ . The OPE with the currents  $W^k(z)$  classify the states in terms of the quantum numbers  $w^k(\alpha)$ ,  $k = 2, 3, \dots, N - 1$ . In particular, the conformal weight is given by

$$w^{(2)}(\alpha) = \Delta(\alpha) = \frac{(\alpha, 2Q - \alpha)}{2}. \quad (3.5)$$

The  $w^k(\alpha)$  are invariant under the  $su(N)$  Weyl group. After a Weyl reflection, the field  $V_\alpha$  acquires a reflection amplitude [37]. As an example, the conjugate representation  $\bar{\alpha}$  is equivalent to  $2Q - \alpha$  under the longest Weyl reflection. This changes the correlation function by a multiplicative factor which will not be relevant to calculate reduced polynomials, as overall factors cancel.

All of this corresponds to the general Toda theory. The  $W_N$  minimal model can be realized as the coset model  $SU(N)_k \oplus SU(N)_1 / SU(N)_{k+1}$ . After imposing the constraint in the root lattice, we find that the primaries are labeled by  $(\rho; \nu) \equiv (\Lambda; \tilde{\Lambda})$  (level  $k$  and  $k+1$  respectively). The details of this construction can be found in [35, 40], for example. As we are going to see below, the  $W_N$  conformal blocks can be obtained by taking the residue of Toda conformal blocks, similarly to the Virasoro and Liouville case.

The Toda 3-point function with one semi-degenerate field was first calculated in [31] and a general formula using AGT relation was proposed in [41, 42]. In the Virasoro case, knowledge of the two and three-point functions allow us to obtain multipoint correlators by the conformal bootstrap [29]. As we saw in section 2.2, the four-point function is completely determined by setting one of the fields to be completely degenerate. However, for the  $W_N$  case this is not enough [30, 31]. The structure of the Verma modules is more constrained by the extra higher-spin symmetries and we need to fix another field to be in a semi-degenerate state  $\alpha = \kappa\omega_{N-1}$ , where  $\kappa$  is an arbitrary constant [37]. Here we shall restrict our discussion to this semi-degenerate case. For more details, see [31, 37, 38].

Three-point correlators are constrained by conformal invariance to be

$$\langle V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)V_{\alpha_3}(x_3) \rangle = \frac{C(\alpha_1, \alpha_2, \alpha_3)}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}}. \quad (3.6)$$

Analysis of the Coulomb integral for calculating this function shows that the structure constants  $C(\alpha_1, \alpha_2, \alpha_3)$  have poles when the screening condition is satisfied

$$(2Q - \sum_{i=1}^3 \alpha_i, \omega_k) = bs_k + b^{-1}\tilde{s}_k, \quad s_k, \tilde{s}_k \in \mathbb{Z}_{\geq 0}. \quad (3.7)$$

Taking the residues of the correlator in those poles gives the  $W_N$  structure constants [35, 37]. Those can be expressed in terms of complicated Coulomb integrals, but in some simple cases, like when one of the fields is semi-degenerate, they can be written in terms of known special functions [37]. Defining

$$C_{\alpha_1, \alpha_2}^{\alpha_3} \equiv C(\alpha_1, \alpha_2, 2Q - \alpha_3), \quad (3.8)$$

our particular case of interest is when one of the fields above is in the fundamental representation

$$C_{-b\omega_1, \alpha_1}^{\alpha_1 - b h_k} = \left( -\frac{\pi\mu}{\gamma(-b^2)} \right)^{k-1} \prod_{i=1}^{k-1} \frac{\gamma(b(\alpha_1 - Q, h_i - h_k))}{\gamma(1 + b^2 + b(\alpha_1 - Q, h_i - h_k))}, \quad (3.9)$$

where  $\gamma(z) = \Gamma(z)/\Gamma(1-z)$ . With this formula, we can show that the fusion rule of  $V_{-b\omega_1}$  and

$V_{\kappa\omega_{N-1}}$  has only two fields. In particular, we have only two intermediate states in a channel with a fundamental and anti-fundamental field and thus the space of conformal blocks is two-dimensional.

Now let us consider the four-point correlator after fixing three points by global  $\text{SL}(2, \mathbb{C})$  invariance

$$\langle V_{-b\omega_1}(z, \bar{z}) V_{\alpha_1}(0) V_{\alpha_2}(\infty) V_{-b\omega_{N-1}}(1) \rangle = |z|^{2b(\alpha_1, \omega_1)} |1-z|^{-\frac{2b^2}{N}} G(z, \bar{z}), \quad (3.10)$$

where  $\alpha_2 = 2Q - \alpha_1$  and, in the  $s$ -channel expansion,

$$G(z, \bar{z}) = \sum_{j=1}^N C_{-b\omega_1, \alpha_1}^{\alpha_1 - bh_j} C(\alpha_1 - bh_j, \alpha_2, -b\omega_{N-1}) G_j(z) G_j(\bar{z}). \quad (3.11)$$

The summation over intermediate states follows from the fusion rules [37, 43]

$$V_{-b\omega_k} V_{\alpha} = \sum_s C_{-b\omega_k, \alpha}^{\alpha - bh_s^{(k)}} \left[ V_{\alpha - bh_s^{(k)}} \right], \quad (3.12)$$

where  $h_s^{(k)}$  are the weights of the representation with highest weight  $\omega_k$ .

The conformal blocks  $G_j(z)$  satisfy the generalized hypergeometric equation

$$\left[ z \prod_{k=1}^N (\theta + A_k) - \theta \prod_{k=1}^{N-1} (\theta + B_k - 1) \right] G_j(z) = 0, \quad (3.13)$$

where  $\theta = z \frac{d}{dz}$  and the coefficients  $A_k$  and  $B_k$  are given by

$$\begin{aligned} A_k &= -b^2 + b(\alpha_1 - Q, e_1 + \cdots + e_{k-1}), \\ B_k &= 1 + b(\alpha_1 - Q, e_1 + \cdots + e_k). \end{aligned} \quad (3.14)$$

In terms of the generalized hypergeometric function,

$$G_1 = {}_N F_{N-1} \left( \begin{matrix} A_1 \dots A_N \\ B_1 \dots B_{N-1} \end{matrix} \middle| z \right), \quad (3.15)$$

$$G_j = z^{1-B_j} {}_N F_{N-1} \left( \begin{matrix} 1-B_j+A_1 \dots 1-B_j+A_N \\ 1-B_j+B_1 \dots 2-B_j \dots 1-B_j+B_{N-1} \end{matrix} \middle| z \right), \quad 1 < j \leq N. \quad (3.16)$$

By consistency between  $s$  and  $u$ -channel expansions, we find

$$\frac{C_{-b\omega_1, \alpha_1}^{\alpha_1 - bh_1} C(\alpha_1 - bh_1, \alpha_2, -b\omega_{N-1})}{C_{-b\omega_1, \alpha_1}^{\alpha_1 - bh_k} C(\alpha_1 - bh_k, \alpha_2, -b\omega_{N-1})} = \frac{\prod_{j=1}^N \gamma(A_j) \gamma(B_{k-1} - A_j)}{\prod_{j=1}^{N-1} \gamma(B_j)} \frac{\prod_{j \neq k-1} \gamma(1 + B_j - B_{k-1})}{\gamma(B_{k-1} - 1)}, \quad (3.17)$$

which follows from connection formulas of generalized hypergeometric functions [37, 44].

Those are the basic equations used to find three-point functions with one partially degenerate field.

#### 4 Crossing and Braiding matrices in $W_N$ models

Now that we know the  $W_N$  conformal blocks, we can analyze particular cases to calculate crossing  $S$  and braiding matrices  $T$  with one fundamental and one anti-fundamental field, as in (3.10). Let us start by taking a highest-weight state in the form

$$\alpha_1 = -b\Lambda - b^{-1}\tilde{\Lambda}, \quad (4.1)$$

related to the pair of representations  $(\Lambda; \tilde{\Lambda})$  labelling a  $W_N$  primary, where

$$\Lambda = \sum_{i=1}^{N-1} n_i \omega_i, \quad \tilde{\Lambda} = \sum_{i=1}^{N-1} \tilde{n}_i \omega_i, \quad n_i, \tilde{n}_i \in \mathbb{Z}_{\geq 0}. \quad (4.2)$$

We are going to see that if a number  $r$  of labels  $n_i$  or  $\tilde{n}_i$  are different from zero, then (3.13) is reducible to a lower order hypergeometric equation. First, notice that

$$B_k = A_{k+1} + b^2 + 1 = A_k + b(\alpha_1, e_k), \quad k = 1, \dots, N-1. \quad (4.3)$$

As  $z(\theta + A)f(z) = (\theta + A - 1)zf(z)$ , it is easy to see that for each  $B_k = A_k$  we can factor out a term  $(\theta + B_k - 1)$  from eq. (3.13), effectively reducing its order. For the particular  $\alpha_1$  we are considering,

$$B_k = A_k - n_k b^2 - \tilde{n}_k. \quad (4.4)$$

Therefore,  $B_k = A_k$  except for  $n_k, \tilde{n}_k \neq 0$ . If  $r$  labels  $n_i$  or  $\tilde{n}_i$  are different from zero, the  $(N, N-1)$  generalized hypergeometric operator factorizes to a product  $D_{N,N-1} = P_{N-r-1} D_{r+1,r}$  of an order  $N-r-1$  operator and a  $(r+1, r)$  hypergeometric operator. This proves our assertion. Finally, we can explicitly write the  $A_k$  as

$$A_k = -(\ell_1 - \ell_k + k)b^2 - (\tilde{\ell}_1 - \tilde{\ell}_k + k - 1), \quad (4.5)$$

where  $\ell_k, \tilde{\ell}_k$  are the number of boxes in the  $k$ -th row of the representation  $\Lambda, \tilde{\Lambda}$ , respectively.

The connection matrices between  $z = 0$  and  $z = 1$  for higher-order hypergeometric equations are not easy to find and we will not consider those in this paper. However, for a reduction to Gauss hypergeometric equation, we have explicit formulas like (3.17). This correspond to the case of *rectangular* representations  $\alpha_1 = -(nb + \tilde{n}b^{-1})\omega_m$ . Here  $m$  corresponds to the number of rows and  $n, \tilde{n}$  the number of columns of the Young diagram of  $\Lambda, \tilde{\Lambda}$  respectively. The field  $\alpha_2 = 2Q - \alpha_1$  is Weyl equivalent to  $\bar{\alpha}_1 = -(nb + \tilde{n}b^{-1})\omega_{N-m}$  and then the correlation function differs from (3.10) by an overall reflection amplitude, which will not be relevant for

us. In this case, eq. (4.5) becomes

$$A_k = -(nH(k-m-1) + k)b^2 - (\tilde{n}H(k-m-1) + k-1), \quad (4.6)$$

where  $H(k)$  is the step function. We have that  $B_k = A_k - (nb^2 + \tilde{n})\delta_k^m$  for  $1 \leq k < N$ , therefore  $B_k = A_k$  for all  $k \neq m$  and (3.13) reduces to a second order hypergeometric equation (2.21) with parameters

$$A = A_m = -mb^2 - (m-1), \quad B = A_N = -(N+n)b^2 - (N+\tilde{n}-1), \quad (4.7)$$

$$C = B_m = -(m+n)b^2 - (m+\tilde{n}-1). \quad (4.8)$$

The  $S$  matrix now is

$$S = \begin{pmatrix} \frac{\Gamma(-(n+m)b^2 - (m+\tilde{n}-1))\Gamma(Nb^2 + N-1)}{\Gamma(-nb^2 - \tilde{n})\Gamma((N-m)(b^2+1))} & \frac{\Gamma(-(n+m)b^2 - (m+\tilde{n}-1))\Gamma(-Nb^2 - (N-1))}{\Gamma(-mb^2 - (m-1))\Gamma(-(N+n)b^2 - (N+\tilde{n}-1))} \\ \frac{\Gamma((m+n)b^2 + m + \tilde{n} + 1)\Gamma(Nb^2 + N-1)}{\Gamma(m(b^2+1))\Gamma((N+n)b^2 + N + \tilde{n})} & \frac{\Gamma((m+n)b^2 + m + \tilde{n} + 1)\Gamma(-Nb^2 - (N-1))}{\Gamma(nb^2 + \tilde{n} + 1)\Gamma(-(N-m)b^2 - (N-m-1))} \end{pmatrix} \quad (4.9)$$

and, as  $S = N_s^{-1}S_{m,n}N_t$ , we have that

$$S = \delta_{m,n} \begin{pmatrix} \xi_{m,n}^2 s_1 & \zeta_{m,n}^2 s_2 \\ \zeta_{m,n}^{-2} s_2 & -\xi_{m,n}^{-2} s_1 \end{pmatrix} = \delta_{m,n} UR S_{m,n} RU^{-1}, \quad (4.10)$$

where

$$R = \begin{pmatrix} \xi_{m,n} & 0 \\ 0 & \xi_{m,n}^{-1} \end{pmatrix}, \quad U = \begin{pmatrix} \zeta_{m,n} & 0 \\ 0 & \zeta_{m,n}^{-1} \end{pmatrix}. \quad (4.11)$$

Relating with the parametrization (2.10), we have that  $\delta_{m,n} = \delta_t/\delta_s$  and

$$N_t = \delta_t RU^{-1} = \delta_t \begin{pmatrix} \gamma_t^{-1} & 0 \\ 0 & \gamma_t \end{pmatrix}, \quad \gamma_t = \frac{\zeta_{m,n}}{\xi_{m,n}}, \quad (4.12)$$

$$N_s = \delta_s R^{-1}U^{-1} = \delta_s \begin{pmatrix} \gamma_s^{-1} & 0 \\ 0 & \gamma_s \end{pmatrix}, \quad \gamma_s = \zeta_{m,n} \xi_{m,n}, \quad (4.13)$$

fixing the normalization in the  $t$  and  $s$ -channel conformal blocks, now defined as  $\mathcal{F}^k = N_k \mathcal{G}^k$ .

Choosing  $\det S_{m,n} = -1$ , the parametrization coefficients are given by

$$\delta_{m,n}^2 = -\det S = \frac{(m+n)b^2 + m + \tilde{n}}{Nb^2 + N - 1}, \quad (4.14)$$

$$\begin{aligned} \zeta_{m,n}^2 &= \left( \frac{S_{12}}{S_{21}} \right)^{\frac{1}{2}} \\ &= \delta_{m,n}^{-1} \sqrt{\frac{[N+n][m]}{[N][n+m]} \frac{\Gamma(m(b^2+1))\Gamma((N+n)b^2+N+\tilde{n})}{\Gamma((N(b^2+1)))\Gamma((m+n)b^2+m+\tilde{n})}}, \end{aligned} \quad (4.15)$$

$$\begin{aligned} \xi_{m,n}^2 &= \left( -\frac{S_{11}}{S_{22}} \right)^{\frac{1}{2}} \\ &= \delta_{m,n}^{-1} \sqrt{\frac{[N][n]}{[N-m][n+m]} \frac{\Gamma(nb^2+\tilde{n}+1)\Gamma(Nb^2+N-1)}{\Gamma((m+n)b^2+m+\tilde{n})\Gamma((N-m)(b^2+1))}}. \end{aligned} \quad (4.16)$$

The orthogonal and symmetric S matrix obtained in (4.10) is thus

$$S_{m,n} = \frac{1}{\sqrt{[m+n][N]}} \begin{pmatrix} (-1)^{m+1} \sqrt{[N-m][n]} & \sqrt{[N+n][m]} \\ \sqrt{[N+n][m]} & (-1)^m \sqrt{[N-m][n]} \end{pmatrix}. \quad (4.17)$$

This is the main result of our paper. For  $N = 2$  and  $n = m = 1$ , we recover the Virasoro case (2.28). Therefore, we conclude that  $S \sim S_{m,n}$  up to a normalization redefinition of the conformal blocks.

Now, to calculate the half-monodromy matrix, let us set  $\tilde{n} = 0$ , as its value only changes monodromy signs and does not change the form of (4.17). The  $s$ -channel braiding matrix is obtained from the asymptotics of the conformal blocks near  $z = 0$ , giving

$$T_{m,n} = N_{m,n}(q) \begin{pmatrix} q^{-n} & 0 \\ 0 & (-1)^m q^m \end{pmatrix}, \quad (4.18)$$

where we choose a framing in which a  $q^{nm/N}$  factor is canceled. Now, consider the case  $n = 1$ . The two special cases in which we can construct knots are the parallel case ( $m = 1$ ) and the anti-parallel case ( $m = N - 1$ ). The framing normalization matrices  $N_{m,n}(q)$  can be chosen using the unknot constraint (2.5) and Yang-Baxter equation (2.6)

$$N_{1,1}(q) = (-1)^{N+1} q^{2-N}, \quad N_{N-1,1}(q) = 1. \quad (4.19)$$

Summing up, the parallel case has the crossing and braiding matrices

$$S = \frac{1}{\sqrt{[2][N]}} \begin{pmatrix} \sqrt{[N-1]} & \sqrt{[N+1]} \\ \sqrt{[N+1]} & -\sqrt{[N-1]} \end{pmatrix}, \quad T = (-1)^N \begin{pmatrix} -q^{1-N} & 0 \\ 0 & q^{3-N} \end{pmatrix}, \quad (4.20)$$



where  $S \equiv S_1$ ,  $T \equiv T_1$ , while the anti-parallel case has

$$\bar{S} = \frac{1}{[N]} \begin{pmatrix} 1 & \sqrt{[N+1][N-1]} \\ \sqrt{[N+1][N-1]} & -1 \end{pmatrix}, \quad \bar{T} = \begin{pmatrix} q^{-1} & 0 \\ 0 & (-1)^{N+1} q^{N-1} \end{pmatrix}, \quad (4.21)$$

where  $\bar{S} \equiv S_{N-1}$  and  $\bar{T} \equiv T_{N-1}$ . For the case of braiding two parallel strands, we can calculate the following invariants

$$ST^p S = \frac{(-1)^{pN} q^{p(2-N)}}{[2][N]} \begin{pmatrix} (-1)^p [N-1] q^{-p} + [N+1] q^p & ((-1)^p q^{-p} - q^p) \sqrt{[N-1][N+1]} \\ ((-1)^p q^{-p} - q^p) \sqrt{[N-1][N+1]} & (-1)^p [N+1] q^{-p} + [N-1] q^p \end{pmatrix}, \quad (4.22)$$

which reduces to (2.32) when  $N = 2$ . Therefore, we can find, for example, the knot polynomial for the trefoil knot  $3_1$ , up to framing redefinition,

$$H_F^{3_1}(q, A) = \frac{(ST^3 S)_{22}}{(STS)_{22}} = q^4 (-1 + A^{-2} (q^2 + q^{-2})). \quad (4.23)$$

We can now try to compare our results with [26] for linking matrices with one link in the fundamental and the other in an arbitrary symmetric ( $m = 1$ ,  $n$  arbitrary) or antisymmetric representation ( $m$  arbitrary,  $n = 1$ ). For appropriate comparison, we note that the quantum dimension of a representation  $R_\lambda$  with partition  $\lambda$  is given by

$$\dim_q R_\lambda = \prod_{(i,j) \in \lambda} \frac{[N+j-i]}{[\ell_i - i + \ell_j^\vee - j + 1]}, \quad (4.24)$$

where  $\ell_j^\vee$  is the number of boxes in the  $j$ -th column of  $\lambda$ . For the antisymmetric case, we get

$$S_{m,1} = \frac{1}{\sqrt{[m+1][N]}} \begin{pmatrix} (-1)^{m+1} \sqrt{[N-m]} & \sqrt{[N+1][m]} \\ \sqrt{[N+1][m]} & (-1)^m \sqrt{[N-m]} \end{pmatrix} \quad (4.25)$$

and

$$S_{N-m,1} = \frac{1}{\sqrt{[N-m+1][N]}} \begin{pmatrix} (-1)^{N-m+1} \sqrt{[m]} & \sqrt{[N+1][N-m]} \\ \sqrt{[N+1][N-m]} & (-1)^{N-m} \sqrt{[m]} \end{pmatrix}. \quad (4.26)$$

These match, up to signs and column permutation, the first and third matrices of sec. 4, item 5 of [26]. To obtain the second matrix, necessary to check the Yang-Baxter equation for links, we have to rederive (3.13) with the fields at  $z$  and at  $z = 0$  interchanged.

For the symmetric case, we get the first and third matrix of sec. 4, item 3 of [26], up to

signs and column permutation,

$$S_{1,n} = \frac{1}{\sqrt{[n+1][N]}} \begin{pmatrix} \sqrt{[N-1][n]} & \sqrt{[N+n]} \\ \sqrt{[N+n]} & -\sqrt{[N-1][n]} \end{pmatrix}, \quad (4.27)$$

and

$$S_{N-1,n} = \frac{1}{\sqrt{[N][N+n-1]}} \begin{pmatrix} (-1)^N \sqrt{[n]} & \sqrt{[N+n][N-1]} \\ \sqrt{[N+n][N-1]} & (-1)^{N-1} \sqrt{[n]} \end{pmatrix}. \quad (4.28)$$

This strongly suggests that (4.17) is the correct crossing matrix for links with one fundamental component and a rectangular component. To calculate the  $T$  matrix in the correct framing, we need the other type of matrix mentioned in [26] to apply the constraints.

## 5 Conclusions

In this work, we have obtained crossing and braiding matrices for certain  $W_N$  algebra representations.  $W_N$  primaries are labelled by two copies of  $SU(N)$  representations and, for the type of correlators described in [37], we did *not* discover any new type of Wilson loop invariants apart from the  $SU(N)_q$  ones. In particular, we can use these matrices to obtain HOMFLY knot invariants in the fundamental representation and two-component HOMFLY link invariants, one component in the fundamental and another in a rectangular representation of  $SU(N)$  algebra. To construct generic link invariants in this case, we need three types of matrices [26], but we have explicitly studied two types, linking and anti-linking. The third (mixed) case can be obtained by making a rederivation of (3.13) with the position of the relevant fields exchanged.

Links with one fundamental component and the other component in an arbitrary  $W_N$  representation are related to the generalized hypergeometric function described in this paper. The connection problem in this case is more intricate but the monodromy group is well known [45]. This is probably enough to find link invariants with one fundamental component linked to an arbitrary higher-representation component, but to get the particular crossing matrices can be more tricky. In principle, the problem of non-trivial multiplicity should be automatically solved by the monodromy properties of the generalized hypergeometric functions. We plan to pursue this approach in a future work.

The case of knot invariants for higher-representations still remains elusive. This is due to the limited knowledge we have about correlation functions in  $W_N$  models, apart from the cases discussed in [37, 38]. It is known that correlators more general than the one with a semi-degenerate field do not obey a linear differential equation [31, 37]. Something can be said about integral representations of more general correlators [38], although via complicated integrals and the monodromy analysis might be interesting for those results. The pentagon identity can, in principle, be used to obtain crossing matrices for knots in higher-representations [14, 26, 28]. However, this approach is computationally expensive from our current knowledge. Promising results have been recently obtained for Toda 3-point functions

and  $W_4$  crossing matrices in [46] with one of the fields in the representation  $\alpha = -b\omega_2$  and the other fields in partially degenerate representations  $\beta_a = k_a\omega_2b$ . Finally, another interesting approach to understand  $W_N$  conformal blocks is via the AGT expansion of isomonodromic tau-functions [47–49]. All of this are relevant lines of attack for the problem of finding Wilson loop invariants for  $W_N$  models. We expect that our results will serve as a basis to further expand the understanding of  $W_N$  models and higher-spin topological invariants in Chern-Simons theory.

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## References

- [1] V. Jones, *Hecke algebra representations of braid groups and link polynomials*, *Annals Math.* **126** (1987) 335–388.
- [2] E. Witten, *Quantum field theory and the jones polynomial*, *Commun.Math. Phys.* **121** (Sept., 1989) 351–399.
- [3] R. K. Kaul and T. R. Govindarajan, *Three-dimensional chern-simons theory as a theory of knots and links*, *Nuclear Physics B* **380** (Aug., 1992) 293–333, [[hep-th/9111063](#)].
- [4] R. Kaul and T. Govindarajan, *Three-dimensional Chern-Simons theory as a theory of knots and links. 2. Multicolored links*, *Nucl.Phys.* **B393** (1993) 392–412.
- [5] P. Ramadevi, T. Govindarajan, and R. Kaul, *Three-dimensional Chern-Simons theory as a theory of knots and links. 3. Compact semisimple group*, *Nucl.Phys.* **B402** (1993) 548–566, [[hep-th/9212110](#)].
- [6] L. Alvarez-Gaume, C. Gómez, and G. Sierra, *Duality and quantum groups*, *Nuclear Physics B* **330** (1990), no. 2 347–398.
- [7] N. Y. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, *Comm. Math. Phys.* **127** (1990), no. 1 1–26.
- [8] A. Mironov, A. Morozov, and A. Morozov, *Character expansion for HOMFLY polynomials. I. Integrability and difference equations*, [arXiv:1112.5754](#).
- [9] A. Mironov, A. Morozov, and A. Morozov, *Character expansion for HOMFLY polynomials. II. Fundamental representation. Up to five strands in braid*, *JHEP* **1203** (2012) 034, [[arXiv:1112.2654](#)].
- [10] A. Mironov, A. Morozov, A. Morozov, P. Ramadevi, and V. K. Singh, *Colored HOMFLY polynomials of knots presented as double fat diagrams*, [arXiv:1504.0037](#).
- [11] A. Mironov, A. Morozov, and A. Morozov, *Evolution method and “differential hierarchy” of colored knot polynomials*, *AIP Conf.Proc.* **1562** (2013) 123–155, [[arXiv:1306.3197](#)].

- [12] A. Anokhina and A. Morozov, *Cabling procedure for the colored HOMFLY polynomials*, *Teor.Mat.Fiz.* **178** (2014) 3–68, [[arXiv:1307.2216](#)].
- [13] D. Galakhov, D. Melnikov, A. Mironov, A. Morozov, and A. Sleptsov, *Colored knot polynomials for arbitrary pretzel knots and links*, *Phys.Lett.* **B743** (2015) 71–74, [[arXiv:1412.2616](#)].
- [14] D.Galakhov, D.Melnikov, A.Mironov, and A.Morozov, *Knot invariants from virasoro related representation and pretzel knots*, [arXiv:1502.0262](#).
- [15] P. Freyd, D. Yetter, J. Hoste, W. R. Lickorish, K. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, *Bulletin of the American Mathematical Society* **12** (1985), no. 2 239–246.
- [16] A. Mironov, A. Morozov, and A. Sleptsov, *Colored HOMFLY polynomials for the pretzel knots and links*, [arXiv:1412.8432](#).
- [17] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, *Nucl. Phys. B* **577** (June, 2000) 419–438, [[hep-th/9912123](#)].
- [18] S. Gukov, A. S. Schwarz, and C. Vafa, *Khovanov-Rozansky homology and topological strings*, *Lett.Math.Phys.* **74** (2005) 53–74, [[hep-th/0412243](#)].
- [19] N. M. Dunfield, S. Gukov, and J. Rasmussen, *The superpolynomial for knot homologies*, *Experimental Mathematics* **15** (2006), no. 2 129–159, [[math/0505662](#)].
- [20] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, *Lett.Math.Phys.* **91** (2010) 167–197, [[arXiv:0906.3219](#)].
- [21] N. Drukker, J. Gomis, T. Okuda, and J. Teschner, *Gauge Theory Loop Operators and Liouville Theory*, *JHEP* **1002** (2010) 057, [[arXiv:0909.1105](#)].
- [22] N. Wyllard,  *$A(N-1)$  conformal Toda field theory correlation functions from conformal  $N = 2$   $SU(N)$  quiver gauge theories*, *JHEP* **0911** (2009) 002, [[arXiv:0907.2189](#)].
- [23] F. Passerini, *Gauge Theory Wilson Loops and Conformal Toda Field Theory*, *JHEP* **1003** (2010) 125, [[arXiv:1003.1151](#)].
- [24] N. Drukker, D. Gaiotto, and J. Gomis, *The Virtue of Defects in 4D Gauge Theories and 2D CFTs*, *JHEP* **1106** (2011) 025, [[arXiv:1003.1112](#)].
- [25] P. Ramadevi, T. Govindarajan, and R. Kaul, *Knot invariants from rational conformal field theories*, *Nucl.Phys.* **B422** (1994) 291–306, [[hep-th/9312215](#)].
- [26] Zodinmawia and P. Ramadevi,  *$SU(N)$  quantum Racah coefficients & non-torus links*, *Nucl.Phys.* **B870** (2013) 205–242, [[arXiv:1107.3918](#)].
- [27] S. Nawata, P. Ramadevi, and Zodinmawia, *Colored HOMFLY polynomials from Chern-Simons theory*, *J.Knot Theor.* **22** (2013) 1350078, [[arXiv:1302.5144](#)].
- [28] J. Gu and H. Jockers, *A note on colored HOMFLY polynomials for hyperbolic knots from WZW models*, [arXiv:1407.5643](#).
- [29] A. Belavin, A. M. Polyakov, and A. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, *Nucl.Phys.* **B241** (1984) 333–380.
- [30] P. Bowcock and G. M. T. Watts, *Null vectors, 3-point and 4-point functions in conformal field theory*, *Theoretical and Mathematical Physics* **98** (Mar., 1994) 350–356, [[hep-th/9309146](#)].

- [31] V. Fateev and A. Litvinov, *On differential equation on four-point correlation function in the conformal toda field theory*, *JETP Lett.* **81** (2005) 594–598; *Pisma Zh.Eksp.Teor.Fiz.* **81** (2005) 728–732, [[hep-th/0505120](#)].
- [32] D.Galakhov, A.Mironov, and A.Morozov, *Wall crossing invariants: from quantum mechanics to knots*, [arXiv:1410.8482](#).
- [33] B. Ponsot and J. Teschner, *Liouville bootstrap via harmonic analysis on a noncompact quantum group*, [hep-th/9911110](#).
- [34] J. Teschner and G. Vartanov, *6j symbols for the modular double, quantum hyperbolic geometry, and supersymmetric gauge theories*, *Lett.Math.Phys.* **104** (2014) 527–551, [[arXiv:1202.4698](#)].
- [35] C.-M. Chang and X. Yin, *Correlators in  $W_N$  Minimal Model Revisited*, *JHEP* **1210** (2012) 050, [[arXiv:1112.5459](#)].
- [36] S. Ribault, *Conformal field theory on the plane*, [arXiv:1406.4290](#).
- [37] V. A. Fateev and A. V. Litvinov, *Correlation functions in conformal Toda field theory I*, *Journal of High Energy Physics* **2007** (Nov., 2007) 002–002, [[arXiv:0709.3806](#)].
- [38] V.A.Fateev and A.V.Litvinov, *Correlation functions in conformal Toda field theory II*, [arXiv:0810.3020](#).
- [39] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal field theory*, .
- [40] P. Bouwknegt and K. Schoutens, *W-symmetry in conformal field theory*, *Physics Reports* **223** (Feb., 1993) 183–276, [[hep-th/9210010](#)].
- [41] V. Mitev and E. Pomoni, *Toda 3-point functions from topological strings*, [arXiv:1409.6313](#).
- [42] M. Isachenkov, V. Mitev, and E. Pomoni, *Toda 3-Point Functions From Topological Strings II*, [arXiv:1412.3395](#).
- [43] V. Fateev and S. L. Lukyanov, *The Models of Two-Dimensional Conformal Quantum Field Theory with  $Z(n)$  Symmetry*, *Int.J.Mod.Phys.* **A3** (1988) 507.
- [44] N. E. Nørlund, *Hypergeometric functions*, *Acta Math.* **94** (Dec., 1955) 289–349. 00072.
- [45] F. Beukers and G. Heckman, *Monodromy for the hypergeometric function  ${}_N F_{N-1}$* , *Invent Math* **95** (June, 1989) 325–354.
- [46] P. Furlan and V. Petkova, *On some 3-point functions in the  $W_4$  CFT and related braiding matrix*, [arXiv:1504.0755](#).
- [47] O. Gamayun, N. Iorgov, and O. Lisovyy, *How instanton combinatorics solves Painlevé VI, V and IIIs*, *J.Phys.* **A46** (2013) 335203, [[arXiv:1302.1832](#)].
- [48] N. Iorgov, O. Lisovyy, and J. Teschner, *Isomonodromic tau-functions from Liouville conformal blocks*, *Commun.Math.Phys.* **336** (2015), no. 2 671–694, [[arXiv:1401.6104](#)].
- [49] P. Gavrylenko, *Isomonodromic  $\tau$ -functions and  $W_N$  conformal blocks*, [arXiv:1505.0025](#).